On the Approximate Nature of the Bivariate Linear Interpolation Function: A Novel Scheme Based on Intensity-Curvature

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Abstract
This paper describes a novel theoretical approach for the improvement of the bivariate linear interpolation function. The fundamental premise of this theory consists of quantifying the effect of the interpolation function on an image’s pixel by the product of the value of the pixel intensity times the sum of non-null second order derivatives of the function. The product is called intensity-curvature term and it is calculated (i) at the grid node, and (ii) at the generic intra-pixel location. The ratio between the two terms consists of the Intensity-Curvature Functional. First order derivatives of the Intensity-Curvature Functional are computed to derive a polynomial system which zeros constitute the Sub-pixel Efficacy Region (SRE). Given a re-sampling location, the SRE is used to project it onto a novel re-sampling location where the approximation properties of the interpolation function lead to error minimization. Two conceptions are thus derived from the Intensity-Curvature Functional: (i) error improvement is set dependent on pixel intensity and curvature of the interpolation function and (ii) the novel re-sampling location varies locally between pixels depending on local properties of the interpolation function as expressed by the intensity-curvature distribution at the neighbourhood. Accordingly, a novel scheme of bivariate linear interpolation is determined with improved approximation properties.

Keywords: Intensity-Curvature Functional, Sub-pixel Efficacy Region, Novel Re-sampling Location.

1. Introduction
Interpolation is a widely used approach in image processing which finds its immediate application in re-sampling a signal at locations where the true value is not known. This is possible by generating a continuous signal estimate and by approximating the signal at the locations of the unknown. Estimation of a continuous signal however produces approximation, which determines the interpolation error. Accuracy of the interpolation functions, that is minimal approximation, was characterized as being affected by the location of the re-sampled points with respect to the initial coordinate system [1-3]. Newton-based descriptions of the interpolation errors are reported in the literature [4-7], and accordingly such errors depend on: (i) the step size (sampling resolution), (ii) the location of the re-sampled point relative to the initial coordinate system (misplacement), and (iii) the value of the interpolation function at the node points. An approach was reported [8, 9] that improves the approximation error of linear interpolation considering a constant shift, and based on it recalculating the coefficients of the interpolator at each node. A recent review of interpolation schemes [10] affirms that approximation properties of interpolation are strictly linked to the sampling resolution.

The basic motivation of the present work is to fill a gap in the current literature. Specifically, to develop an interpolation scheme that has its approximation property dependent on the joint information content determined by (i) the node intensity and (ii) the second order partial derivative of the interpolation function, and that, consequently to the developmental approach, makes re-sampling locally variable pixel by pixel. To accomplish this goal, the theory outlined in this paper starts from the intuition to embed curvature of the interpolation function and pixel intensity into the same formulation. This is determined by means of two intensity-curvature terms, which ratio is called the Intensity-Curvature Functional ($\Delta E$). Study of the polynomial of first order derivatives of $\Delta E$ reveals the existence of a region within each pixel, referred to as Sub-pixel Efficacy Region (SRE), that can be used to compute novel re-sampling locations in order to calculate the interpolation function with improved approximation. The theory shows that the spatial location of the SRE is dependent on the relationships between the intensity values at the pixel to re-sample and the neighbourhood, and the local curvature of the interpolation function. It implies that re-sampling is determined locally variable that is pixel-by-pixel. Local re-sampling is not new however, since it has been
2. Theory

The basic tool of the theory is grounded by a mathematical formulation which enables the improving of approximation properties for the bivariate linear interpolation function. This formulation is called Intensity Curvature Functional and it is dependent on the local properties of the discrete signal and the curvature of the function.

2.1 Definition of Intensity-Curvature Functional

Let \( h \) be the bivariate linear interpolation function: \( h(x, y) \) as defined in [13] (its formulation is reported in the appendix). Let the interpolation function and its derivatives be continuous within the pixel [14]. Given an original pixel intensity value \( f(0,0) \), to re-sample at a location \( (x, y) \) determines a novel pixel intensity \( h(x, y) \) which is different from the original and is dependent on the curvature of the interpolation function. The curvature of the function, characterized by local convexity or concavity, is a mean of differentiation between shapes and behaviour within the neighbourhood. Let the curvature be expressed by the sum of the second order partial derivatives:

\[
\left( \frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right)
\]

This sum is incorporated into the two intensity-curvature terms, which are defined as the integral of the interpolation function times the sum of its non-null second order partial derivatives. While the intensity-curvature term calculated at the generic intra-pixel location \( (x, y) \neq (0,0) \) is termed \( E_{o}(x, y) \), the intensity-curvature term calculated at the grid point \( (x, y) = (0, 0) \) is termed \( E_{0}(x, y) \). The information content about the function behaviour in the neighbourhood is therefore increased as opposed to considering the intensity values alone. For a given pixel, let the intensity-curvature term calculated at a generic intra-pixel location \( (x, y) \neq (0,0) \) be defined as:

\[
E_{IN}(x, y) = \int \int h(x, y) \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right) dx \; dy
\]

and let the intensity-curvature term calculated at the grid point \( (x, y) = (0, 0) \) be defined as:

\[
E_{0}(x, y) = \int \int f(0,0) \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right) dx \; dy
\]

Calculation of (1) and (2) lead to:

\[
E_{IN}(x, y) = 2 H_{xy}(x, y) \omega_f
\]

\[
E_{0}(x, y) = 2 x y f(0,0) \omega_f
\]

Where:

\[
H_{xy}(x, y) = \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right)_{(0,0)} = 2 \omega_f
\]

\( H_{xy}(x, y) \) is the primitive function of \( h(x, y) \) with respect to \( y \) and \( x \). \( \omega_f \) is defined (see appendix) as that value which expresses relationships between \( f(0,0) \), intensity at the pixel to re-sample, and its neighboring pixel intensity values \( f(1,0), f(0,1) \) and \( f(1,1) \). The formulation of \( E_{0}(x, y) \) and \( E_{IN}(x, y) \) cover the cases for two given pixel intensity values, with their difference being either in terms of intensity values, or in terms of second order partial derivatives, or both. The Intensity-Curvature Functional is defined as the ratio between intensity-curvature terms that is:

\[
\Delta E(x, y) = \frac{E_{0}(x, y)}{E_{IN}(x, y)}
\]

Because of the ratio between terms, \( \Delta E \) measures the pixel intensity-curvature change caused by the interpolation function and, because each of the two terms is calculated by integration, it also measures the intensity-curvature distribution across the pixel.

2.2 Sub-pixel Efficacy Region

Geometrically, the Sub-pixel Efficacy Region (SRE) is the set of points within the pixel by which is possible to calculate novel re-sampling locations, where the local curvature of the interpolation function is such to estimate a value that gives minimal approximation. The novel re-sampling locations are determined by the use of the extreme points of the Intensity-Curvature Functional \( \Delta E \), which constitute the Sub-pixel Efficacy Region. Therefore, the Intensity-Curvature Functional \( (\Delta E) \) will be calculated by solving the polynomial system of its first order derivatives where the extreme points of \( \Delta E \) will be found, and thus relationships between values of intensity at neighboring pixels and misplacement will be derived. These will be used to compute the novel re-sampling locations in order to calculate the interpolation function.
2.2.1 Definition

For the bivariate liner interpolation function, the Sub-pixel Efficacy Region is defined as:

\[ \Phi = \left\{ (x, y) : \frac{\partial (\Delta E(x, y))}{\partial x} = 0, \frac{\partial (\Delta E(x, y))}{\partial y} = 0 \right\} \]  

(6)

2.2.2 Approach

Calculation of first order derivatives of \( \Delta E \) yields:

\[ \frac{\partial (\Delta E(x, y))}{\partial x} = \frac{\alpha (x, y)}{4 H_{xy}^2} \]  

(7)

\[ \frac{\partial (\Delta E(x, y))}{\partial x} = \frac{\alpha (x, y)}{4 H_{xy}^2} \]  

(8)

\[ \gamma(x, y) = 2 \left( \frac{xy}{2} \right)^2 f(0, 0) \]  

(9)

\[ \alpha(x, y) = \gamma(x, y) \left[ (f(1, 0) - f(0, 0)) + \omega r \frac{y}{2} \right] \]  

(10)

\[ \alpha(y, x) = -\gamma(x, y) \left[ (f(0, 1) - f(0, 0)) + \omega r \frac{x}{2} \right] \]  

(11)

Four extremes are found by equating (7) and (8) to zero and solving the resulting polynomial system by use of a Groebner basis [15]:

\[ P_1 \equiv [0, 0] \]

\[ P_2 \equiv \left[ 0, -\frac{2 (f(1, 0) - f(0, 0))}{\omega r} \right] \]

\[ P_3 \equiv \left[ -\frac{2 (f(0, 1) - f(0, 0))}{\omega r}, 0 \right] \]

\[ P_4 \equiv \left[ -\frac{2 (f(0, 1) - f(0, 0))}{\omega r}, -\frac{2 (f(1, 0) - f(0, 0))}{\omega r} \right] \]

Points \( P_1 \) through \( P_4 \) determine the Sub-pixel Efficacy Region \( \Phi \). The following vertex is considered in the remainder of the text:

\[ x_{sre} = \frac{-2 (f(0, 1) - f(0, 0))}{\omega r} \]  

(12)

\[ y_{sre} = \frac{-2 (f(1, 0) - f(0, 0))}{\omega r} \]  

(13)

Once the Sub-pixel Efficacy Region is identified, in order to re-sample the signal at the misplacement \( (x_0, y_0) \), let \( (x^{\text{r}}, y^{\text{r}}) \) be the novel re-sampling location. The latter is derived as follows. The product of the interpolation function times its second order partial derivative is considered at the location \( (x_{sre} - x^{0}, y_{sre} - y^{0}) \) that is:

\[ H_1 = \left\{ h(x_{sre}, y_{sre}) \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} \right)(x_{sre}, y_{sre}) \right\} \]

(14)

The product of the interpolation function times its second order partial derivative is also calculated at the location \( (x_{sre}, y_{sre}) \) that is:

\[ H_2 = \left\{ h(x_{sre}, y_{sre}) \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} \right)(x_{sre}, y_{sre}) \right\} \]

(15)

And the ratio between \( H_1 \) and \( H_2 \) is equated to:

\[ \frac{H_1}{H_2} = \Delta E^* \]  

(16)

The novel re-sampling location is determined from (14) as:

\[ x^{r} = \frac{\eta_{xy} - \mu_{x}}{\lambda_{x}} \]  

(17)

\[ y^{r} = \frac{\eta_{xy} - \mu_{y}}{\lambda_{y}} \]  

(18)

Where:

\[ \rho = (f(0, 0) + x_{sre} \theta_{x} + y_{sre} \theta_{y} + x_{sre} y_{sre} \omega r) \omega r \]

(19)

\[ \eta_{xy} = [\rho \Delta E^* - f(0, 0)] \]

(20)

\[ \mu_{x} = [(y_{sre} - y_{0}) \theta_{y} + x_{sre} (\theta_{x} + (y_{sre} - y_{0}) \omega r)] \]

(21)

\[ \mu_{y} = [(x_{sre} - x_{0}) \theta_{x} + y_{sre} (\theta_{y} + (x_{sre} - x_{0}) \omega r)] \]

(22)

\[ \lambda_{x} = [(y_{0} - y_{sre}) \omega r - \theta_{x}] \]

(23)

\[ \lambda_{y} = [(x_{0} - x_{sre}) \omega r - \theta_{y}] \]

(24)

And \( \theta_{x}, \theta_{y} \) are defined in the appendix.
3. Results

3.1 Approach

To quantify the advantages of the Sub-pixel Efficacy Region it is necessary to show the potential improvement that can be obtained with re-sampling using the novel locations furnished by (15) and (16). For the simulations presented below, it is assumed that each pixel has the origin of the coordinate system located at the grid point. The procedure adopted is as follows. An image is calculated at intra-pixel location \( (x_0, y_0) \). The image is then motion corrected and interpolated and the root mean square error (RMSE) is computed. The image is also motion corrected and interpolated of the novel re-sampling location obtained from (15) and (16). The RMSE is calculated, and is shown to be smaller than the RMSE obtained by motion correction and interpolation of the misplacement \( (x_0, y_0) \).

Demonstrations are given with a 2D image (“Lena”), Magnetic Resonance Imaging (MRI), and functional MRI of the human brain. Furthermore, it is shown that extending both theory and methods to the quadratic B-Spline also leads to improvement of the B-Spline interpolation error. Values of misplacement that are reported throughout text and figures are pure numbers, with the intent that the behaviour of the RMSE for varying misplacement is seen as a lookup table for corresponding values of sampling resolution. Except for the B-Spline, values of misplacements \((0.01, 0.01)\) through \((0.09, 0.09)\) are indicated in the figures as \((0.01, 0.01)\) through \((0.09, 0.09)\).

3.2 Simulations

Three functions, \( \sin(x, y) \), \( \cos(x, y) \) and \( \ln(x, y) \) were calculated on a grid of a \( 32 \times 32 \) matrix for \( (x, y) \in [0.1, 3.2] \times [0.1, 3.2] \). Each function was also calculated at misplacements \((x_0, y_0)\) of \((0.01, 0.01)\) through \((0.09, 0.09)\) at steps of \((0.01, 0.01)\). These images were motion corrected of the misplacement \((x_0, y_0)\), and of the novel re-sampling location obtained from (15) and (16), and interpolated in either case using the bivariate linear function. Also, in either case, the RMSE was computed by subtracting the resulting image from the original image. Figure 1 shows the original image in (a) for the \( \sin(x, y) \) function. The plot of \( \Delta E \) versus misplacement as computed using the bivariate linear interpolation function is shown in (b) and (c) respectively with images that were calculated at the intra-pixel location \((x_0, y_0)\), and images that were motion corrected and interpolated of \((x_0, y_0)\). Figure 1 also shows the plot of the values of \( \Delta E \) in (d) and \( \Delta E \) in (e), which resulted from (12) and (13) respectively. Figures 1b and 1c show one of the spatial points of the Sub-pixel Efficacy Region (SRE) by the white vertical line. At the SRE location, the Jacobian of second order derivatives was negative across the \( 32 \times 32 \) grid, showing that the \( \Delta E \) was a maximum. Plots of \( \Delta E \) in figure 1b quantify the difference in terms of intensity-curvature, between the original image and the image calculated at \((x_0, y_0)\), and in figure 1c, between the original image and the image obtained after motion correction and interpolation of \((x_0, y_0)\).

Figure 1. (a) \( \sin(x, y) \) in its original form. Plots of \( \Delta E \) versus misplacement: before motion correction (b), and after motion correction (c). The white vertical line indicates the location of the point \( P_1 \) of the Sub-pixel Efficacy Region (SRE). Plot of the SRE coordinates: \( x \) (d) and \( y \) (e), and their distribution across the grid. Values were obtained from (12) and (13) and normalized in \([0, 1]\).

Figure 2. RMSE obtained from \( \sin(x, y) \) using the bivariate linear function for motion correction and interpolation of the misplacement \((x_0, y_0)\) (black), and of the novel re-sampling location \((x^*, y^*)\) obtained from (15) and (16) (white).

Figure 3. Plot of novel re-sampling locations across the \( 32 \times 32 \) grid for \( \cos(x, y) \) with a misplacement \((x_0, y_0)\) - \((0.01, 0.01)\). Values of \( x \) in (a) and of \( y \) in (b) were obtained respectively from (15) and (16).

Figures 1d and 1e demonstrate how the location of the extreme of \( \Delta E \) varies pixel by pixel. Figure 2 shows the
sin (x, y) function and the RMSE curves obtained from images that were motion corrected and interpolated of the misplacement (x₀, y₀) (black), and from images that were motion corrected and interpolated of the novel locations as per (15) and (16) (white). The RMSE reduction demonstrates the interpolation improvement obtained for the bivariate linear function because of the use of the Sub-pixel Efficacy Region (SRE). For a misplacement of (0.01, 0.01), figure 3 shows the behaviour of the novel re-sampling locations for sin (x, y) in the plot of the values of x and y obtained respectively from (15) and (16). That is, instead of using a misplacement of (0.01, 0.01) at each pixel location, the novel re-sampling location, which varies pixel by pixel, is used for motion correction and interpolation with the bivariate linear function. Simulations described in this section were performed also for cos (x, y) and ln (x, y) functions, and similar results were obtained.

3.3 The Effect of the Sub-Pixel Efficacy Region

The simulations performed reveal that the calculation of the novel re-sampling location using (15) and (16) leads to better approximation, and is thus established that the point (x_{sre}, y_{sre}) of the Sub-pixel Efficacy Region is sufficient to obtain interpolation improvement. Its necessity was tested using random numbers differing across pixels in place of (x_{sre}, y_{sre}) and calculating the novel re-sampling location. Both higher RMSE values and random fluctuations in the RMSE curve’s behaviour were obtained. The point (x_{sre}, y_{sre}) assumes the role of reference by means of which, it is possible to calculate the novel re-sampling location. When the ratio:

\[ \frac{H_1}{H_2} \]

of equation (14) is closest to the value of one, the products of intensity times curvature at both the numerator and denominator are the almost the same. Therefore to justify the use of:

\[ \Delta E^* = \frac{E_{IN}(x_{sre} \cdot x_0 \cdot y_{sre} \cdot y_0)}{E_{IN}(x_{sre} \cdot y_{sre})} \]

in the place of one and in the right term of (14), it needs to be shown that the interpolation improvement is higher when using \( \Delta E^* \). When removing \( \Delta E^* \) from (14), based on the novel re-sampling locations that were subsequently calculated, higher values of RMSE were obtained. This latter is illustrated in table I where the RMSE sum is calculated for the sin (x, y) function across misplacements of (0.01, 0.01) through (0.09, 0.09) at steps of (0.01, 0.01).

Table I. For the sin(x, y) function, from left to right column: RMSE obtained by classic bivariate interpolation (bef. Impr.), by SRE-based bivariate interpolation using \( \Delta E^* \) in (14) (\( \Delta E^* \)), and using the value of one in place of \( \Delta E^* \) (One). Lower values of RMSE were obtained with SRE-based bivariate interpolation using \( \Delta E^* \).

In summary: (i) \( \Delta E \) is a measure of intensity-curvature change, (ii) extremes of \( \Delta E \) locates the Sub-pixel Efficacy Region, and (iii) using the properties of the SRE it is possible to compute the novel re-sampling locations and to obtain improvement of interpolation error. The use of (15) and (16) as locations for calculating the bivariate linear function will be referred to as SRE-based bivariate linear interpolation in order to differentiate with the classic bivariate linear interpolation which uses the misplacement (x₀, y₀).

3.4 Results with Real Image Data

The results presented here were obtained with: “Lena”, MRI and functional MRI of the human brain. “Lena” consisted of a matrix of 128 x 128 pixels, subsampled from an image consisting of an original matrix of 208 x 222. MRI resolution is also 128 x 128 with 2.0 x 2.0 mm of pixel size, and 18 slices covering the entire volume with inter-slice resolution of 6.0 mm. Each slice was sub-sampled from an original matrix of 256 x 256. Functional MRI resolution is 64 x 64 with 3.75 x 3.75 mm of pixel size, with sixteen slices and an inter-slice resolution of 6.00 mm. To show reduction in the interpolation error, the paradigm adopted was the same as the one adopted for the simulations. Image data was shifted at steps of 0.01 along x and y simultaneously, or rotated at steps of 0.01 deg, and in either case within the

![Figure 4. Lena image: (a) original; (b) inverted RSE images obtained after motion correction and interpolation of the misplacement (0.07, 0.07) in the upper row, and of the rotation 0.07 deg in the lower row. Left: obtained by classic bivariate linear interpolation, and right: by SRE-based interpolation. (c) Behaviour of RMSE across the range of misplacements and rotations: classic bivariate linear interpolation (black), and SRE-based bivariate linear interpolation (white).]
range [0, 1.1]. The images were then motion corrected and interpolated with classic and with SRE-based bivariate linear interpolation functions, and in either case the RMSE was calculated. No threshold was applied to the images such to take into account all pixel intensity values. Figure 4 shows: (a) the original 2D “Lena” image, (b) from left to right, inverted root square error (RSE) images obtained with both classic and with SRE-based bivariate linear interpolation for a misplacement of (0.07, 0.07) (upper row), and a rotation of 0.07 (lower row). Noticeable is the RSE improvement showing an almost zero residual in the case of the SRE-based interpolation. Figure 4c shows the behaviour of the RMSE in the range of misplacements in [0, 1.1] and rotations in [0, 1.1 deg]. The white lines, indicating SRE-based interpolation, show consistently a smaller RMSE across the range.

Figure 5 shows sample results for the MRI: (a) an original MRI slice, (b) from left to right, inverted RSE images obtained after motion correction and interpolation by classic bivariate of (0.46, 0.46), (0.55, 0.55), and 0.46 deg. In the lower row, inverted RSE images were obtained after motion correction and SRE-based interpolation. The SRE-based interpolation led to overall improved estimation. Figures 5c and 5d show the behaviour of the sum of RMSE across the 18 slices of the MRI volume for misplacements in [0, 1.1] (c), and rotations in [0, 1.1 deg] (d). Similarly to MRI, results indicate SRE-based interpolation error improvement obtained with low resolution functional MRI data. Figures 5e and 5f show results for misplacements in [0, 1.1] and rotations in [0, 1.1 deg]. In figures 5c, 5d, 5e, and 5f, the RMSE obtained with SRE-based interpolation is shown in white and the RMSE obtained with classic bivariate linear interpolation is shown in black.

3.5 Quadratic B-Spline
The theory and methodology outlined by this research have been extended to other interpolation paradigms. With the details of the mathematical foundation for this work considered to be beyond the scope of this paper, this section focuses on the simulations for the validation completed for cos (x) of the one-dimensional quadratic SRE-based B-Spline interpolation. The classic B-Spline function used has a 3 x 2 neighbourhood and it is reported in [16] as:

\[
h_3(x) = -2a|x|^2 + \frac{(a+1)}{2} \\
0 \leq |x| \leq 1/2
\]

\[
a|x|^2 - \frac{3(a+1)}{4} |x|^{3/2} \\
1/2 \leq |x| \leq 3/2
\]

The cos (x) function was calculated at 32 nodes in the range [0.1, 3.2] and also calculated at misplacements within [0.01, 0.09] at steps of 0.01 in the x direction. The constant term “a” was chosen to be -0.87 in (23).
an important objective of this work to ascertain, and thus determine qualitatively, how results obtained by the SRE-based bivariate linear interpolation compare to the classic form of parametric quadratic B-Spline given in (23). In order to do so, a misplacement of 0.07 along x and a value of the constant “a”=1.14 were used for motion correction and interpolation of “Lena” and a sagittal MRI slice, by the use of the B-Spline (23) as shown in figure 6b, and by the use of the SRE-based bivariate linear interpolation function as shown in figure 6c. Subtraction revealed a quasi-null difference, as it can also be seen in the figure by visual inspection. This result suggests that, at least for the one-dimensional case and for small values of misplacement, the use of the SRE-based linear interpolation permits approximation properties of the order of the B-Spline (23). Consistent results were obtained with MRI and functional MRI data.

Figure 6. (a) Plot of the eRMSE for B-Spline interpolation of cos (x). The RMSE line obtained by the quadratic SRE-based B-Spline (white) indicates improvement of the interpolation error with respect to the RMSE line obtained by classic quadratic B-Spline (black). (b) Images obtained by classic quadratic B-Spline (FH3 – No SRE), and (c) images obtained by SRE-based bivariate linear interpolation (FL2 – SRE).

4. Discussion and Conclusions

4.1 Literature

Quantitatively, two general types of approaches are reported to attempt exact interpolation. One is that of solving linear systems to find optimized coefficients of the interpolator [17] and the other is to obtain the coefficients of the interpolator by convolution [3, 18, 19]. It is also important to note that, within the context of finding optimized coefficients, osculatory interpolation [20], which has shown [21] to be equivalent to more recent convolution-based approaches [22], presents schemes which optimality depend on the choice of free parameters. In general, the difference between true value and interpolation function estimate decreases when the sampling step becomes smaller and so does the interpolation error [6, 10, 17]. What differentiates this work from what previously reported in literature is that instead of optimized constants [17, 23], or fixed shifts [8, 9], the methodology presented here determines (15) and (16) on that basis obtains novel re-sampling locations in order to calculate the interpolation function, locally and dependently from a shift that varies from pixel to pixel. Indeed, results presented by this research point out that interpolation approximation can be improved on the basis of the joint information content of pixel intensity and local curvature of the interpolation function. The Intensity-Curvature Functional AE has been used to reveal that the locations where to re-compute the sampling point vary across pixels because of the intensity-curvature variation across pixels. Therefore, the consideration of re-sampling as a local issue rather than a global one, improves the approximation properties of the interpolation function. In this regard, the present research shows a similarity to the work described in [12], which recalculates nodes’ location as dependent on the derivatives of the interpolation function at the original grid placement. In contrast, this novel scheme re-computes locations based on the Sub-pixel Efficacy Region and in a way that they are dependent on the pixel intensity, and the function second order partial derivatives calculated at (i) the original grid placement and at (ii) the given re-sampling location (x0, y0). As far as the validation methodology, non-contextual experiments as those based on the sine function were not exhaustive and thus the validation was extended to contextual evaluation of “Lena”, MRI, and functional MRI volumes. Although this paper addresses linear interpolation, which is not as accurate as the B-Spline, comparing the SRE-based bivariate linear interpolation function with the quadratic B-Spline favours the potentiality of the theory herein presented. Finally, the validation approach that was used, which consisted of applying a transformation followed by the inverse, might be considered as having limiting characteristics since such approach has shown to be not as efficient in revealing interpolation errors as it is instead repeatedly applying the transformation and returning the image to the initial grid position without interpolation [9].

4.2 The Novel Theory

Adaptive linear interpolation methods have been devised with the aim to estimate missing pixels from images based on neighbouring pixel intensity and Euclidean distances [24, 25]. A more exhaustive review of linear interpolation paradigms that use scattered data points, and not a uniform grid, is given in [25]. Within this context, the novel theory proposed here assumes relevance to the extent that considers re-sampling as a local issue. Also, it is relevant to mention here the improved linear paradigm recently proposed in [8, 9], based on which the interpolation function is calculated at a constant shift, whereas in the present approach the shift varies from pixel to pixel depending on the local curvature of the interpolation function. To further elaborate this concept, and also to clarify the nature of the Sub-pixel Efficacy Region, as the mean by which it is possible to calculate the interpolation function with minimized detrimental effects, figure 7 illustrates how the novel re-sampling location is obtained. When re-
sampling at the location \((x_0, y_0)\), the interpolation surface \(\beta = h(x, y)\) determines an approximation that exists because of the difference between the locations: (i) at the plane \(\alpha\), which would be ideally the surface to approximate and (ii) at the surface \(\beta = h(x, y)\). That is, at the location \((x_0, y_0)\) the surface \(\beta = h(x, y)\) differs from the plane \(\alpha\) which it aims to approximate of the quantity \(\lambda\). On the other hand, the approximation \(\lambda\) tends to be nullified at the novel re-sampling location \((x^0, y^0)\) due to the local curvature of \(\beta\), which makes the surface \(\beta\) almost tangential to \(\alpha\). From this illustration, one can conclude that in order to locally calculate the novel re-sampling location, a mathematical instrument is needed in order to assimilate, within one single formulation, both the intensity and the local curvature. Such instrument has been called Intensity-Curvature Functional \((\Delta \text{E})\). Within this context, it is also important to explain the meaning of the \((x_{\text{sre}}, y_{\text{sre}})\) location, which is one of the points of the Sub-pixel Efficacy Region.

**Figure 7. Conceptualization of the Sub-pixel Efficacy Region.** At a misplacement \((x_0, y_0)\), the approximation \(\lambda\) is determined by the difference between the interpolation surface \(\beta = h(x, y)\) and the plane \(\alpha\) that is being approximated by \(\beta\). The approximation \(\lambda\) tends to zero, as the local curvature is such to make \(\beta\) tangential to \(\alpha\).

The \((x_{\text{sre}}, y_{\text{sre}})\) location is used in (14) as a reference point to determine the projection of \((x_{\text{sre}} - x_0, y_{\text{sre}} - y_0)\) onto \((x_{\text{sre}} - x^0, y_{\text{sre}} - y^0)\), such to determine the novel location \((x^0, y^0)\). To do so, (14) requires that the ratio between intensity times curvature at \((x_{\text{sre}} - x^0, y_{\text{sre}} - y^0)\), and \((x_{\text{sre}}, y_{\text{sre}})\), is equal to the ratio:

\[
\Delta \text{E}^* = \frac{E_{\text{IN}}(x_{\text{sre}} - x_0, y_{\text{sre}} - y_0)}{E_{\text{IN}}(x_{\text{sre}}, y_{\text{sre}})}
\]

This ratio, in (14), is the projection factor that is applied to:

\[
\hat{h}(x_{\text{sre}}, y_{\text{sre}}) \left( \frac{\partial^2 h(x, y)}{\partial x \partial y} \right)_{(x_{\text{sre}}, y_{\text{sre}})}
\]

which is the product of intensity times the curvature at \((x_{\text{sre}}, y_{\text{sre}})\). Therefore, \((x_{\text{sre}}, y_{\text{sre}})\) projects \((x_0, y_0)\) to determine the novel location \((x^0, y^0)\). It is reasonable to infer that it is questionable to re-sample the signal at a location that is not exactly the same as the misplacement \((x_0, y_0)\). But the justification is guaranteed by the benefits of the application of the theory, which is the reduction of the interpolation error measured as root-mean-square-error. This is so because the novel re-sampling location where to calculate the interpolation function determines improved approximation properties. For the presentation of the theory, the \(P_4\) vertex of the Sub-pixel Efficacy Region has been evaluated. It is possible however to obtain improvements also with the other two vertices: \(P_1\) and \(P_2\).

### 4.3 Practical Implications of the Theory

This paragraph offers an overview of practical implications of the theory. In particular, it is informative to see how the theory can be proved useful in the estimation of signals of unknown nature, for which it does not exist any set of samples at given intra-node locations in time, because of the sampling frequency of the equipment used for recording. That is for samples that cannot be recorded by the sampling equipment because of the constraints imposed by the Nyquist theorem. In order to determine whether or not the novel re-sampling location obtained through the SRE ensures an improved estimation with respect to the given re-sampling location, one should devise an experiment in the following manner. Let a one dimensional signal be given which value we know at any location \(m_0\) of its domain. The quadratic B-Spline interpolation model of (23) is chosen to re-sample the signal at \(m_0\). If we are able to demonstrate that the signal is re-sampled by the quadratic SRE-based B-Spline function with increased accuracy with respect to the classic paradigm of (23), we could certainly infer that the former is of value and use in the estimation of signals of unknown nature, and thus its estimation should be more accurate than that of the classic scheme. Thus, the following experimentation has been conducted. At a given re-sampling location \(m_0\) the quadratic B-Spline given in (23) was used to re-sample the signals \(\sin(x)\) and \(\ln(x)\). Further, the signal was calculated at the novel re-sampling location \(m^0\) obtained on the basis of \(m_0\). This is equivalent to re-sampling using the quadratic SRE-based B-Spline (the computation to derive the value of \(m^0\) and thus to calculate the quadratic SRE-based B-Spline is deemed beyond the scope of this paper). Therefore, given that the signal is known at any location, it is possible to ascertain the accuracy of the two interpolation models by the use of R = \((1 - \text{RMSE}_{\text{B-SPLINE}}) / \text{RMSE}_{\text{SRE-B-SPLINE}}\) ratio of improvement; where \(\text{RMSE}_{\text{B-SPLINE}}\) and \(\text{RMSE}_{\text{SRE-B-SPLINE}}\) were calculated respectively with classic and SRE-based quadratic B-Spline functions. This experimentation was conducted with sampling resolution of 0.01, with \(m_0 \in [0, 0.011]\) increasing from 0 at steps of 0.0001 for 111 steps. The value of R was computed and plotted in figure 8 versus the re-sampling location \(m_0\). Negative values of R show that the quadratic SRE-based B-Spline furnishes improved capability in the estimation of the true value of the signal, with respect to the classic quadratic B-Spline of (23), consistently across the all range of \(m_0\). Similar results were obtained with the \(\cos(x)\) function and with sampling resolutions of 0.1 and 0.001.
Appendix

A.1 Definition of $h(x,y)$

Let $h$ be a continuous function that takes the form:

$$h(x,y) = f(0,0) + x(f(1,0) - f(0,0)) + y(f(0,1) - f(0,0)) + xy(f(1,1) + f(0,0) - f(0,1) - f(1,0))$$  \hspace{1cm} (A-1)$$

Where $f(0,0)$, $f(1,0)$, $f(0,1)$ and $f(1,1)$ are the values of intensity at the four corners of the pixel. $h(x,y)$ is the bivariate linear interpolation function. Let be

$$\theta_x = (f(1,0) - f(0,0)), \quad \theta_y = (f(0,1) - f(0,0))$$

$$\omega_r = (f(1,1) + f(0,0) - f(0,1) - f(1,0)).$$

It follows that:

$$h(x,y) = f(0,0) + x\theta_x + y\theta_y + xy\omega_r$$  \hspace{1cm} (A-2)$$

A.2 Primitives of $h(x,y)$

On the basis of (A-2), it follows that the primitive of $h(x,y)$ with respect to $y$ that is $H_y$, and with respect to $x$ that is $H_x$, and with respect to $y$ and $x$ that is $H_{xy}$ are:

$$H_y(x,y) = f(0,0)y + yy\frac{\theta_y}{2} + xy\frac{\omega_r}{2}$$  \hspace{1cm} (A-3)$$

$$H_x(x,y) = f(0,0)x + xx\frac{\theta_x}{2} + x^2\frac{\omega_r}{2}$$  \hspace{1cm} (A-4)$$

$$H_{xy}(x,y) = H_{yx}(x,y) = f(0,0)xy +$$

$$\frac{yx^2\theta_x}{2} + \frac{xy^2\theta_y}{2} + \frac{x^2y^2\omega_r}{4}$$  \hspace{1cm} (A-5)$$

It is also true that:

$$H_y = \frac{\partial (H_{xy}(x,y))}{\partial x}, \quad H_x = \frac{\partial (H_{xy}(x,y))}{\partial y}$$

A.3 Second order derivatives of $\Delta E(x,y)$

The determinant $\Sigma^{(2d)}$ [27] was calculated to ascertain the nature of the extremes of $\Delta E$:

$$\Sigma^{(2d)} = \begin{vmatrix} \frac{\partial^2 (\Delta E(x,y))}{\partial x^2} & \frac{\partial^2 (\Delta E(x,y))}{\partial x \partial y} \\ \frac{\partial^2 (\Delta E(x,y))}{\partial y \partial x} & \frac{\partial^2 (\Delta E(x,y))}{\partial y^2} \end{vmatrix}$$  \hspace{1cm} (A-6)$$

Calculation of the second order derivatives of $\Delta E$ leads to the following expressions:
\[ \frac{\partial^2 (\Delta E(x, y))}{\partial x^2} = 2 \rho_{xy} x y^2 \tau_{xy} H_{xy}(x, y) \Theta_{xy} \]  
(A - 7)

\[ \frac{\partial^2 (\Delta E(x, y))}{\partial y^2} = 2 \rho_{xy} x^2 y \tau_{xy} H_{xy}(x, y) \Theta_{xy} \]  
(A - 8)

\[ \frac{\partial^2 (\Delta E(x, y))}{\partial \xi \partial \eta} = 2 \rho_{xy} x^2 \kappa_{xy} H_{xy}(x, y) \Omega_{xy} \]  
(A - 9)

\[ \frac{\partial^2 (\Delta E(x, y))}{\partial \xi \partial \eta} = 2 \rho_{xy} y^2 \kappa_{xy} H_{xy}(x, y) \Omega_{xy} \]  
(A - 10)

Where:

\[ \Theta_{xy} = \frac{2 H_{xy}(x, y) - x H_x(y, y)}{v_{xy}} \]  
(A - 11)

\[ \Theta_{yx} = \frac{2 H_{xy}(y, x) - y H_y(x, y)}{v_{xy}} \]  
(A - 12)

\[ \Omega_{yx} = \frac{y^2 \text{or} \left( \frac{3}{2} H_{xy}(x, y) - 2 y H_x(x, y) \right)}{v_{xy}} \]  
(A - 13)

\[ \Omega_{xy} = \frac{x^2 \text{or} \left( \frac{3}{2} H_{xy}(x, y) - 2 x H_y(x, y) \right)}{v_{xy}} \]  
(A - 14)

\[ \rho_{xy} = 2 f(0,0) \omega_{f}^4 \]  
(A - 15)

\[ \tau_{xy} = \left[ f(1,0) - f(0,0) + \frac{\omega_{r} y}{2} \right] \]  
(A - 16)

\[ \tau_{yx} = \left[ f(1,0) - f(0,0) + \frac{\omega_{r} x}{2} \right] \]  
(A - 17)

\[ \kappa_{y} = 2 y f(1,0) \]  
(A - 18)

\[ \kappa_{x} = 2 x f(0,1) \]  
(A - 19)

\[ v_{xy} = \left[ 2 H_{yx}(x, y) \omega_{f} \right]^4 \]  
(A - 20)

6. References


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