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On the Approximate Nature of the Bivariate Linear Interpolation Function: A Novel Scheme Based on Intensity-Curvature

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Abstract

This paper describes a novel theoretical approach for the improvement of the bivariate linear interpolation function. The fundamental premise of this theory consists of quantifying the effect of the interpolation function on an image's pixel by the product of the value of the pixel intensity times the sum of non-null second order derivatives of the function. The product is called intensity-curvature term and it is calculated (i) at the grid node, and (ii) at the generic intra-pixel location. The ratio between the two terms consists of the Intensity-Curvature Functional. First order derivatives of the Intensity-Curvature Functional are computed to derive a polynomial system which zeros constitute the Sub-pixel Efficacy Region (SRE). Given a re-sampling location, the SRE is used to project it onto a novel re-sampling location where the approximation properties of the interpolation function lead to error minimization. Two conceptions are thus derived from the Intensity-Curvature Functional: (i) error improvement is set dependent on pixel intensity and curvature of the interpolation function and (ii) the novel re-sampling location varies locally between pixels depending on local properties of the interpolation function as expressed by the intensity-curvature distribution at the neighbourhood. Accordingly, a novel scheme of bivariate linear interpolation is determined with improved approximation properties.

Keywords: *Intensity-Curvature Functional, Sub-pixel Efficacy Region, Novel Re-sampling Location.*

1. Introduction

Interpolation is a widely used approach in image processing which finds its immediate application in re-sampling a signal at locations where the true value is not known. This is possible by generating a continuous signal estimate and by approximating the signal at the locations of the unknown. Estimation of a continuous signal however produces approximation, which determines the interpolation error. Accuracy of the interpolation functions, that is minimal approximation, was characterized as being affected by the location of the re-sampled points with respect to the initial coordinate system [1-3]. Newton-based descriptions of the interpolation errors are reported in the literature [4-7], and accordingly such errors depend on: (i) the step size (sampling resolution), (ii) the location of the re-sampled point relative to the initial coordinate system (misplacement), and (iii) the value of the interpolation function at the node points. An approach was reported [8, 9] that improves the approximation error of linear interpolation considering a constant shift, and based on it recalculating the coefficients of the interpolator at each node. A recent review of interpolation schemes [10] affirms that approximation properties of interpolation are strictly linked to the sampling resolution.

The basic motivation of the present work is to fill a gap in the current literature. Specifically, to develop an interpolation scheme that has its approximation property dependent on the joint information content determined by (i) the node intensity and (ii) the second order partial derivative of the interpolation function, and that, consequently to the developmental approach, makes re-sampling locally variable pixel by pixel. To accomplish this goal, the theory outlined in this paper starts from the intuition to embed curvature of the interpolation function and pixel intensity into the same formulation. This is determined by means of two intensity-curvature terms, which ratio is called the Intensity-Curvature Functional (ΔE). Study of the polynomial of first order derivatives of ΔE reveals the existence of a region within each pixel, referred to as Sub-pixel Efficacy Region (SRE), that can be used to compute novel re-sampling locations in order to calculate the interpolation function with improved approximation. The theory shows that the spatial location of the SRE is dependent on the relationships between the intensity values at the pixel to re-sample and the neighbourhood, and the local curvature of the interpolation function. It implies that re-sampling is determined locally variable that is pixel-by-pixel. Local re-sampling is not new however, since it has been explored within the context of optimization approaches aimed at re-calculating node position of B-Spline models and is reported in the literature [11, 12].

The remaining parts of the paper are organized as follows. The next section outlines the theory, defines the formulation called Intensity-Curvature Functional, and by its examination, the Sub-pixel Efficacy Region is revealed. The section on results presents simulations and validation with real data to demonstrate the improvement of the interpolation approximation. Also, an example is given to demonstrate the potentiality of the theory and methods to be extended to Spline interpolation. Finally, in the conclusion section, the work done here is discussed within the context of the most current literature and future efforts.

2. Theory

The basic tool of the theory is grounded by a mathematical formulation which enables the improving of approximation properties for the bivariate linear interpolation function. This formulation is called Intensity Curvature Functional and it is dependent on the local properties of the discrete signal and the curvature of the function.

2.1 Definition of Intensity-Curvature Functional

Let h be the bivariate linear interpolation function: h = h(x, y) as defined in [13] (its formulation is reported in the appendix). Let the interpolation function and its derivatives be continuous within the pixel [14]. Given an original pixel intensity value f(0, 0), to re-sample at a location (x, y) determines a novel pixel intensity h(x, y) which is different from the original and is dependent on the curvature of the interpolation function. The curvature of the function, characterized by local convexity or concavity, is a mean of differentiation between shapes and behaviour within the neighbourhood. Let the curvature be expressed by the sum of the second order partial derivatives:

$$\left(\frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right)$$

This sum is incorporated into the two intensity-curvature terms, which are defined as the integral of the interpolation function times the sum of its non-null second order partial derivatives. While the intensity-curvature term calculated at the generic intra-pixel location (x, y) ≠ (0, 0) is termed $E_{IN}(x, y)$, the intensity-curvature term calculated at the grid point (x, y) = (0, 0) is termed $E_o(x, y)$. The information content about the function behaviour in the neighbourhood is therefore increased as opposed to considering the intensity values alone. For a given pixel, let the intensity-curvature term calculated at a generic intra-pixel location (x, y) ≠ (0, 0) be defined as:

$$E_{IN}(x, y) = \int_0^x \int_0^y h(x, y) \left(\frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right) dx\, dy \quad (1)$$

and let the intensity-curvature term calculated at the grid point (x, y) = (0, 0) be defined as:

$$E_o(x, y) = \int_0^x \int_0^y f(0, 0) \left(\frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right)_{(0, 0)} dx\, dy \quad (2)$$

Calculation of (1) and (2) lead to:

$$E_{IN}(x, y) = 2\, H_{xy}(x, y)\, \omega_f \quad (3)$$

$$E_o(x, y) = 2\, x\, y\, f(0, 0)\, \omega_f \quad (4)$$

Where:

$$\left(\frac{\partial^2 h(x, y)}{\partial x \partial y} + \frac{\partial^2 h(x, y)}{\partial y \partial x} \right)_{(0, 0)} = 2\, \omega_f$$

$H_{xy}(x, y)$ is the primitive function of h(x, y) with respect to y and x. ω_f is defined (see appendix) as that value which expresses relationships between f(0, 0), intensity at the pixel to re-sample, and its neighboring pixel intensity values f(1,0), f(0,1) and f(1,1). The formulation of $E_o(x, y)$ and $E_{IN}(x, y)$ cover the cases for two given pixel intensity values, with their difference being either in terms of intensity values, or in terms of second order partial derivatives, or both. The Intensity-Curvature Functional is defined as the ratio between intensity-curvature terms that is:

$$\Delta E(x, y) = \frac{E_o(x, y)}{E_{IN}(x, y)} \quad (5)$$

Because of the ratio between terms, ΔE measures the pixel intensity-curvature change caused by the interpolation function and, because each of the two terms is calculated by integration, it also measures the intensity-curvature distribution across the pixel.

2.2 Sub-pixel Efficacy Region

Geometrically, the Sub-pixel Efficacy Region (SRE) is the set of points within the pixel by which is possible to calculate novel re-sampling locations, where the local curvature of the interpolation function is such to estimate a value that gives minimal approximation. The novel re-sampling locations are determined by the use of the extreme points of the Intensity-Curvature Functional ΔE, which constitute the Sub-pixel Efficacy Region. Therefore, the Intensity-Curvature Functional (ΔE) will be calculated by solving the polynomial system of its first order derivatives where the extreme points of ΔE will be found, and thus relationships between values of intensity at neighboring pixels and misplacement will be derived. These will be used to compute the novel re-sampling locations in order to calculate the interpolation function.